Homework K

- **Due: Tuesday, March 9, before class**
- Please refer to the syllabus for homework policies. (Points of interest include late policies, typesetting, collaboration, extra-credit stuff...)
- If you explicitly do not want your submission to be considered as a solution key, please state so clearly at the top of your submission. If you explicitly want to remain anonymous (if selected), please state so clearly at the top of your submission.
- If you use dynamic programming in your solution, please address the steps outlined in the typed notes.
- There are also some multiple-choice/short-answer problems on Gradescope.
- There are many wonderful problems on max flow and applications of max flow to be found in Kleinberg-Tardos and Jeff's notes that are a good warmup for the problem here.
- There is no need to include your PUID on the submission thanks to Gradescope.

Problems

1. Exercise 2 in the typed notes (about densest subgraph).

   We realized that this problem is too difficult without some substantial hints:

   (a) Show that $g(S) \geq \lambda|S|$ is equivalent to

   $$m \geq \lambda|S| + \sum_{u \in V \setminus S} \frac{\deg(u)}{2} + \frac{|\partial(S)|}{2}.$$

   Here $\partial(S)$ is the cut induced by S, anad $\deg(u)$ is the degree of a vertex u. m is the number of edges in the graph. [1]

   (b) Consider the following auxiliary graph. Starting from the given graph $G = (V, E)$, assign capacity $1/2$ to all the edges. Add a source vertex s and add a directed edge (s, v) with weight $\deg(v)/2$ for all $v \in V$. Add a sink vertex t and a directed edge (v, t) with capacity λ for all $v \in V$. Consider the (s, t)-cuts in the graph. The cut $\partial(s)$ has capacity m. What about other (s, t)-cuts of the form $\partial(s \cup \{R\})$, where $R \subset V$?

[1] Even if you can't show this, you may solve the rest of the problem assuming the above is true for partial credit.